Searching periodic orbits with a modulation based on Shannon entropy

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It is known that a variety of stable periodic orbits can be created if a weak periodic perturbation is applied to a deep chaotic state. However, which orbit can be created is unpredictable. Because the periodicity of a time series can be determined by a Shannon entropy, we suggest that these orbits can be distinguished by this entropy and we may find the desired periodic orbit if we can further modify the perturbation guiding ourselves by this entropy. Thus, the method may provide a goal-oriented scheme for taming chaos with a weak periodic perturbation. We show theoretically the scheme in three different models, namely, a modified logistic mapping, a driven Duffing-Holmes oscillator, and a directly modulated semiconductor laser. The influence of noise is discussed and the efficiency of searching is estimated.

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I. INTRODUCTION

Many potentially useful ideas, such as synchronization chaos and controlling chaos, have been developed in nonlinear dynamics particularly in the past few years. Practical implementations of these ideas in engineering and other technological fields have been of general interest and importance [1]. Indeed, the presence of chaos can be a great advantage because a variety of periodic motions can be stabilized in a chaotic attractor as emphasized by Ott, Grebogi, and Yorke (OGY) [2]. Their method has been a classical means in the field of controlling chaos [3]. As a feedback method, in response to the dynamics of the system the OGY method may require a fast feedback mechanism that may be very complicated. On the other hand, by introducing a weak periodic perturbation, Braiman and Goldhirsch (BG) theoretically illustrate a simple nonfeedback scheme for creating stable periodic orbits as an alternative way of controlling chaos [4]. Their method is very attractive for its simple implementation. Unfortunately a serious weakness exists in the BG method, i.e., what kind of orbits can be created is not known at all [3]. Because the BG method is so simple, any possible solution of the above problem would be of importance. This paper will explore this possibility theoretically. To have an answer, a close look at the controlling scheme that is based on a weak periodic perturbation would be helpful. Our point is that if we can recognize the status of output wave forms and then enforce the system toward the desired output region based on such a recognition, we may overcome the shortcoming so that a goal-oriented controlling scheme should be brought out.

The essential point is how to recognize the status of an output wave form, particularly the periodicity. In the BG method, the maximum Lyapunov exponent is used to identify the status of a time series. If chaos is suppressed, the exponent is negative. On the other hand, a direct observation of time series (or return map) also can help us. However, a direct observation of time series makes an automatic experiment (simulation) difficult or even impossible. Also a calculation of the Lyapunov exponent requires a long-time series. Nevertheless, the development of nonlinear dynamics has brought us several available quantities for this task. In this paper, we adopt the Shannon entropy as an alternative choice of characterization. To the people of nonlinear dynamics, Shannon entropy is used for the characterization of the ergodicity of the dynamics. Then one may think an evaluation of Shannon entropy also requires a long time series. However, to determine the periodicity, it is not necessary to use a long time series as shown below. Since we can further modify the perturbation, guided by the Shannon entropy we may find the desired periodic orbit. In the following we will present our searching scheme in detail. We will demonstrate the proposed scheme in three different models including a modified logistic mapping, a driven Holmes-Duffing oscillator, and a directly modulated semiconductor laser. By these examples, the efficiency of searching as well as the influence of noise will be discussed.

II. OUTLINE OF THE SEARCHING SCHEME

We first summarize our searching procedure.

(1) Choose a physical quantity for observation, say $X(t)$; pick up the values of N successive maxima, i.e., $\{X_i\}(i=1,2,3,\ldots,N)$ after transient.

(2) Sort these N maxima by their values and count the number of $\{X_i\}$ appearing in each group, say N_j $(j=1,2,\ldots,M)$. (We assume that these N maxima are divided into M groups.) After this, calculate the probability of the lth group P_l by dividing N_l with the total number $N(l=1,2,\ldots,M)$. We determine the Shannon entropy by these P_l.

(3) Guided by the value of Shannon entropy, we introduce (or modify) an external weak periodic perturbation to search the desired periodic orbit. Next, we give the detail. The first point is: *why maxima*. The reasons are as follows.

(1) It can be understood that the acquisition of successive maxima from a time series converts a continuous time series into a discrete one. In a real situation, the finite sampling rate in data acquisition also leads to a

discrete time series.

(2) Not less common, we pick up the maxima of a continuous time series since a peak detector is available even in hardware.

(3) A maximum requires a zero time derivative, i.e., $dX(t)/dt=0$, so such a sampling will form a special Poincaré section.

(4) The time difference between two successive maxima shows the time scale inherent in the system and reveals some physical characteristics of the system.

(5) By such a definition, we obtain a faster way in calculating the entropy in comparison with previous works [5].

The second point is: *what kind of Shannon entropy.* The Shannon entropy H is defined as

$$H=-\sum_{l=1}^{M} P_l \log_2 P_l \ \text{(unit:bit)} , \tag{1}$$

where P_l and N are defined as in procedure (2) shown above. This Shannon entropy represents the average information coded in the data by the system [6]. Here it serves as an indicator of periodicity. The entropy value of an arbitrary wave form with periodicity up to 6 is shown in Fig. 1 where the typical wave form within one cycle is shown in the box. A period-1 pulse will be with $H=0$ while an irregular wave form shows a large H. Note that a period-m orbit can be further distinguished. Furthermore, one can set the conditional statement in the program to meet a particular value of the Shannon entropy for searching the desired periodic region. Because a

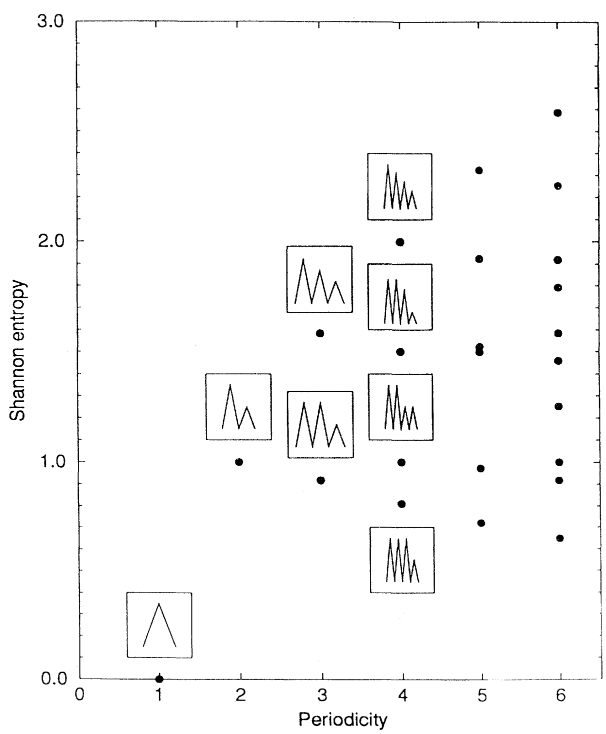

FIG. 1. The values of Shannon entropy at different periodicities. A wave form within one circle is shown in the box for illustration (see text).

complicated wave form has a large H, we can use the Shannon entropy rather than the maximum Lyapunov exponent to indicate the complexity of waveforms. For searching the desired periodic orbit this way is rather efficient. This leads to our third point: *why the Shannon entropy.* It should be noted that if a nonchaotic state is chosen initially, then only simple dynamic behavior is observed. This answers the concern: *why chaos is necessary.* The appearance of chaos is very important to the BG method. Chaos is a very complex structure in phase space and it is so sensitive that a variety of periodic motions may be created by a weak periodic perturbation.

Indeed, the entropy here is a resolution-dependent quantity. A resolution parameter is needed to classify these maxima since even in numerical simulations to have exactly the same two maximum values is difficult. This may not be a defeat. In turn, the resolution used here provides a simple way of handling the effect of noise. Since the proposed scheme is supposed to be implemented in real systems, its dependence on noise should be of concern. Since it is a searching method, it is demanding to estimate the efficiency of searching. In Sec. III, we will illustrate these points by examples.

We make a short summary and outline the possible extension. The method does not require model equation in advance. In real cases, one can measure the Shannon entropy to indicate the complexity of the wave form. At first, one can take three to four trials to determine the entropy in a nearby region, and drive the system toward the region of low entropy. One can repeat this procedure and gradually guide the system away from chaos and toward the desired periodic region. By this guidance, we would not need to search the whole parameter space. Also because it is the periodic region to be searched, one only needs to accumulate some data for calculating the entropy. Such a searching scheme should be easy to implement in experiments and can be automated. We will provide our experimental work in the near future.

III. DEMONSTRATION

To demonstrate the proposed scheme, we use a modified logistic mapping as the first model. The model mapping is as follows:

$$x_{n+1}=r_1 x_n(1-x_n)-\lambda y_n+\eta\xi_n , \tag{2}$$

$$y_{n+1}=r_2 y_n(1-y_n) . \tag{3}$$

Here, Eq. (3) is a logistic mapping and is only used for the generation of a stable periodic signal. For Eq. (2), a fully developed chaos occurs at $r_1=4.0$ when we set the perturbation intensity $\lambda=0$ and the noise intensity $\eta=0$ [7]. In Eq. (2), ξ_n is a number within [0,1] which is chosen from a uniform random number generator. After transient, the y_n shows a stable period-2 signal at $r_2=3.33$. Let us use this simple period-2 signal as a perturbation. The effect of periodic perturbation is shown in Fig. 2(a) where only $50x_n$ ($n=3950$–4000) are used for making the bifurcation diagram. One can see that the fully developed chaos is suppressed and transformed into vari-

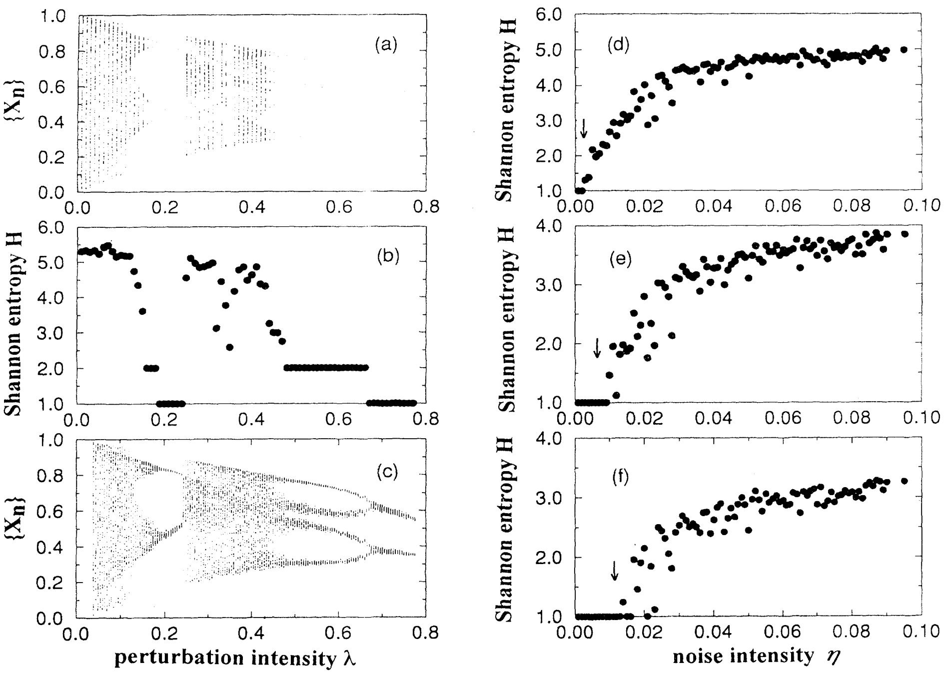

FIG. 2. (a) and (b) A bifurcation diagram: the x_n vs the perturbation intensity λ for a modified logistic mapping and the corresponding Shannon entropy diagram. (c) A bifurcation diagram: the noise added x_n vs the perturbation intensity λ. (d)–(f) The entropy value of a period-2 orbit vs the noise intensity η with the resolution 0.01, 0.03, and 0.05, respectively.

ous dynamic states for different λ. The corresponding diagram of Shannon entropy is shown in Fig. 2(b). (Here the maxima are simply the x_n and only 50 data are used to evaluate the entropy.) To explore the dependence of noise, we set $\eta \neq 0$. By Fig. 2(c) where $\eta = 0.01$, one can see the change. Though this noise-induced change does affect our searching scheme, by adjusting the resolution range this effect can be reduced. As an illustration, for $\lambda = 0.7$, we vary the noise intensity from 0 to 0.1 and calculate the corresponding Shannon entropy. Figures 2(d), 2(e), and 2(f) are for the resolution ranges with 0.01, 0.03, and 0.05, respectively. [As a note, the resolution range used in Fig. 2(b) is less than 10^{-8}.] Indeed, for a smaller resolution range, the periodic orbit cannot be well recognized and even leads to failure. This can be seen clearly by the regime with η larger than the value indicated by the down arrow ↓. This special value indicates the upper limit of the tolerance of noise for the characterization scheme proposed here.

Next we present another demonstration using a model of the Duffing-Holmes oscillator [8],

$$\frac{d^2}{dt^2}y + \delta\frac{d}{dt}y - 0.5y(1-y^2) = f\cos(\omega t) + r_1 f\cos(r_2\omega t)\ , \qquad (4)$$

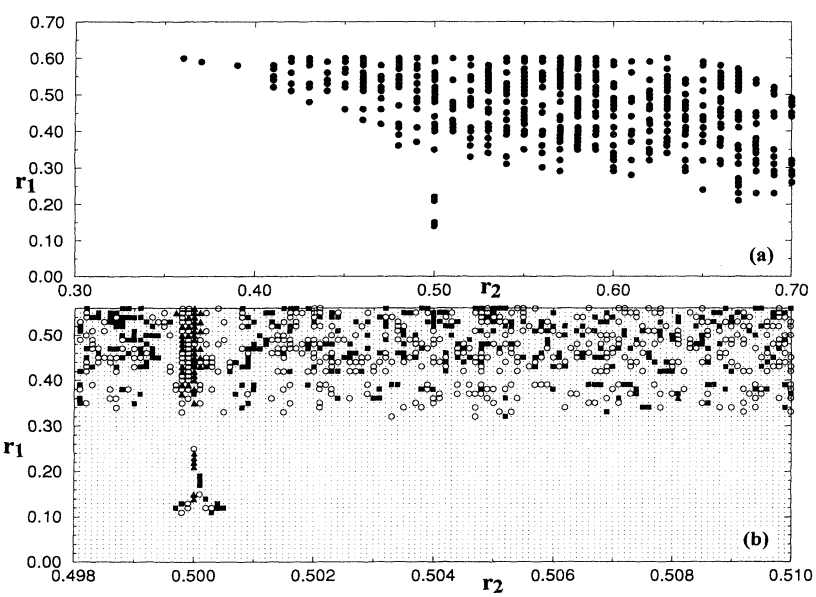

FIG. 3. (a) A two-dimensional state diagram with a maximum Lyapunov exponent for the Duffing-Holmes oscillator where r_1 and r_2 are the amplitude index and frequency index of the weak perturbation signal, respectively (see text). The filled circle point means that the corresponding exponent is negative. (b) The two-dimensional state diagram in terms of Shannon entropy where the filled triangle is for $H(s) \leq 3$, the empty circle means $3 < H(s) \leq 4$, and the filled square corresponds $4 < H(s) \leq 4.5$. We leave $H(s) > 4.5$ as the dotted point.

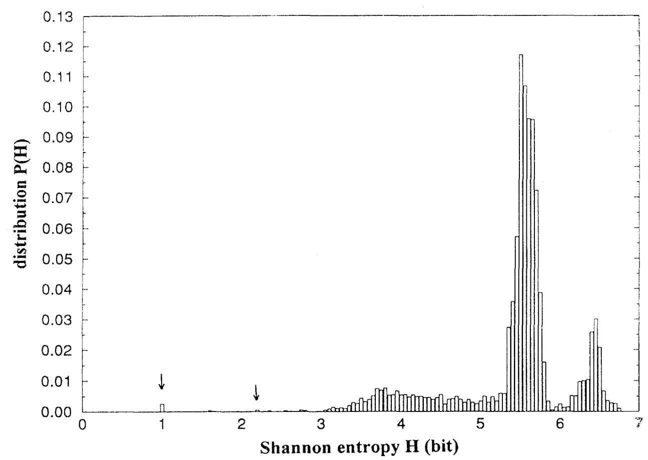

FIG. 4. The entropy distribution diagram of 7200 perturbed states for the Duffing-Holmes oscillator. This distribution is obtained based on Fig. 3(b).

where $\delta=0.15$ and $\omega=0.8$ and $f=0.23$. If $r_1=0$, by this parameter setting the oscillator shows chaos. Next we vary r_1 from 0.3 to 0.7 and r_2 from 0 to 0.6. (The total number of perturbed states is 7200.) The maximum Lyapunov exponent is determined as shown in Fig. 3(a) where the region with the negative maximum Lyapunov exponent is shown with dotted points. We take a smaller region and calculated the Shannon entropy to represent the dynamics. (Here 700 data of maximum points are used to determine the entropy.) In such a two-dimensional phase diagram of entropy, various symbols are used to show different entropy value ranges. One can see that a variety of dynamic states has been created, [see Fig. 3(b)]. Next for this Fig. 3(b), we count the occurrence of perturbed states at various entropy values. By dividing the total number of states (here 7200), we derive a distribution diagram as shown in Fig. 4. Here only a few low-period orbits indicated by the down arrows ↓ can be found. Most of the orbits have larger entropy values. One can realize that the proposed searching scheme would not succeed if the desired periodic orbit cannot be created. If in the beginning we set the perturbation (i.e., r_1 and r_2) randomly, how quickly can we find the desired orbit? Since the probability of finding the desired orbit is proportional to the area occupied by the orbit, Fig. 4 may provide us with the efficiency of searching.

As a final example, we take a semiconductor laser model [9] for the technical interest. Usually this model is rewritten in terms of two normalized, dimensionless densities N and P for a theoretical discussion and the set of rate equations is as follows:

$$\tau_e \frac{d}{dt} N = \frac{I}{I_{\mathrm{th}}} - N \frac{N-\delta}{1-\delta} P \; , \tag{5}$$

$$\tau_p \frac{d}{dt} P = \frac{N-\delta}{1-\delta}(1-\varepsilon P)P - P + \beta N \; , \tag{6}$$

$$I(t) = I_b + I_m \sin(2\pi f_m t) + r_1 I_m \sin(2\pi r_2 f_m t) \; , \tag{7}$$

where δ and ε are two dimensionless parameters, I_{th} is the threshold current of the semiconductor laser, I is the injection current, τ_e and τ_p are the electron and photon lifetime respectively, and β is the spontaneous emission factor. To generate high-speed pulse signals, we usually modulate the injection current in the form of Eq. (7) with $r_1=0$, where I_b is the bias current, I_m is the modulation current, and f_m is the modulation frequency. In simulation, we set $I_b/I_{\mathrm{th}}=1.5$, $\tau_p=6\mathrm{ps}$, $\tau_e=3$ ns, $\beta=5.0\times10^{-5}$, $\delta=0.692$, and $\varepsilon=1.0\times10^{-4}$ as in Ref. [9]. For $I_m/I_{\mathrm{th}}=0.55$ and $f_m=0.8$ GHz, chaos appears if $r_1=0$.

The temporal characteristics of the laser model in a

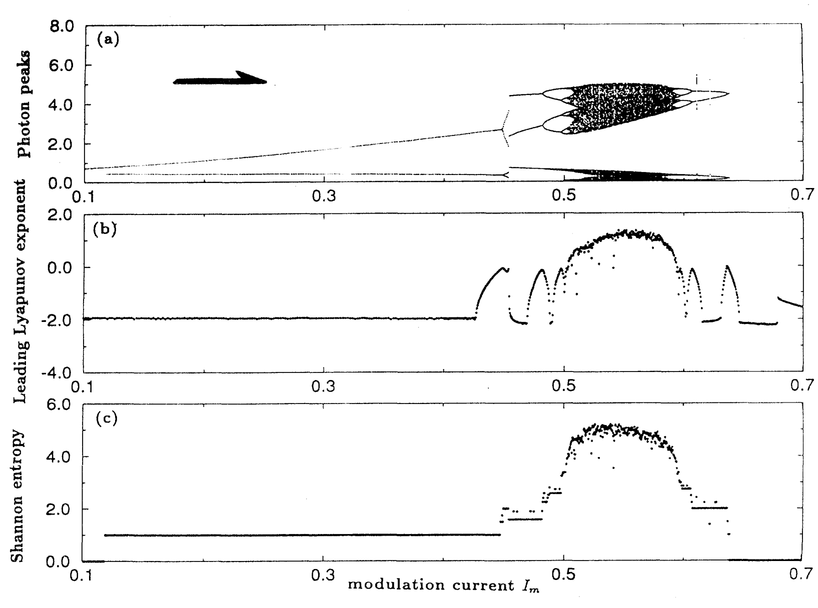

FIG. 5. Dynamic features of a directly modulated semiconductor laser model. (a) The bifurcation diagram of the peaks of photon density (photon peaks) vs the swept modulation current I_m. The direction of the arrow shows the sweeping direction. (b) The corresponding diagram in terms of the leading Lyapunov exponent. (c) The corresponding diagram of the Shannon entropy. The parameters are $I_b/I_{\mathrm{th}}=1.5$, $\tau_p=3$ ps, $\tau_e=3$ ns, $b=5.0\times10^{-5}$, $\delta=0.692$, $\varepsilon=1.0\times10^{-4}$, and the modulation frequency $f_m=0.8$ GHz.

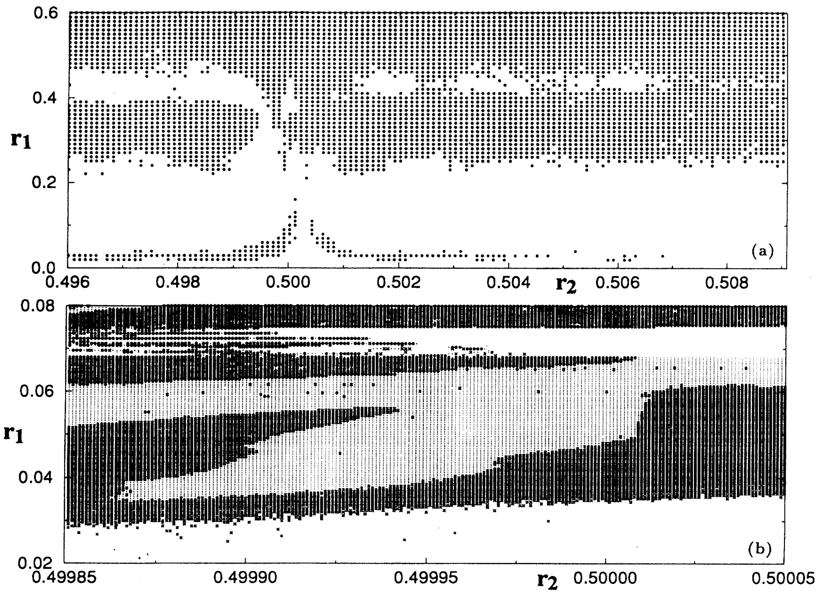

FIG. 6. Two-dimensional state diagram with a second modulation index of amplitude r_1 and frequency r_2 as controlled variables. In (a) the dotted region is with a negative Lyapunov exponent; in (b) the filled circle points are of $H(P)\leq 2$, the dotted represents $2<H(P)\leq 2.5$, the crossed shows $2.5<H(P)\leq 3$, and the black points are for $H(P)>3$.

wide range of parameters can be described by a bifurcation diagram in terms of the peak photon density (photon peaks [9]) as shown in Fig. 5(a) where $f_m=0.8$ GHz. For simplicity, we only present the simulation that occurs along the forward sweeping direction of modulation current in Fig. 5(a). (The direction is indicated by the arrow $\rightarrow$.) The phase of the maximum Lyapunov exponent is shown in Fig. 5(b). It can be seen that even for the period-1 pulses, different maximum Lyapunov exponents are possible. A time series with a negative maximum Lyapunov exponent does not mean that its wave form is simple. Thus a measure of the Lyapunov exponent cannot tell us how close we are to a desired periodic orbit. We reinterpret the bifurcation diagram in Fig. 5(c) in terms of the Shannon entropy. (Here 250 data of maximum points are used.) One can see the periodicity clearly. Next by varying the amplitude and frequency of the second modulation r_1 and r_2, we can obtain a phase diagram as shown in Fig. 6(a) where the dotted region has a negative maximum Lyapunov exponent. The appearance of dotted regions indicates the suppression of chaos. Figure 6(b) shows the corresponding entropy diagram. It can be understood that there are many paths for arriving at the periodic region we want. To estimate the efficiency of searching, we derive at a distribution diagram. (The total number of perturbed states is 40 000). As seen by Fig. 7, a low-period orbit (such as a period-1 pulse) is difficult to create here. However, some low-period orbits do exist. In Fig. 8, we show how taming chaos can be achieved based on a weak periodic perturbation. Typical chaotic spiking is shown in Fig. 8(a). After taming with a weak perturbation, periodic spiking is created as shown in Fig. 8(b).

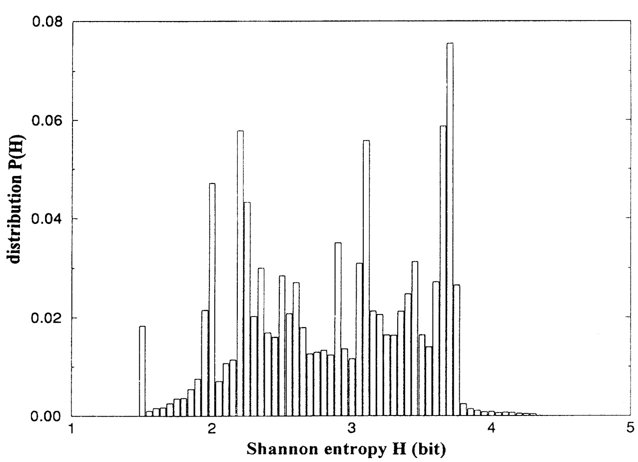

FIG. 7. The entropy distribution diagram of 40 000 perturbed states for a directly modulated semiconductor laser. This distribution is obtained based on Fig. 6(b).

IV. CONCLUSION

In summary, a simple scheme for searching the desired periodic orbits in chaos has been shown. We have exercised our scheme in three different models. The effect of noise and the searching efficiency have been discussed. Our scheme provides a simple way toward a goal-oriented scheme of taming chaotic dynamics with a weak periodic perturbation.

It should be emphasized that the controlling scheme that is based on a weak periodic perturbation is very efficient and simple. With the help of Shannon entropy, one can further improve it. This success is rooted in the choice of entropy as a measure of the periodicity and a relational parameter in the conditional statement of searching program. Certainly, if the desired orbit cannot be created, the searching scheme proposed here does not work. The efficiency of searching can be estimated by the

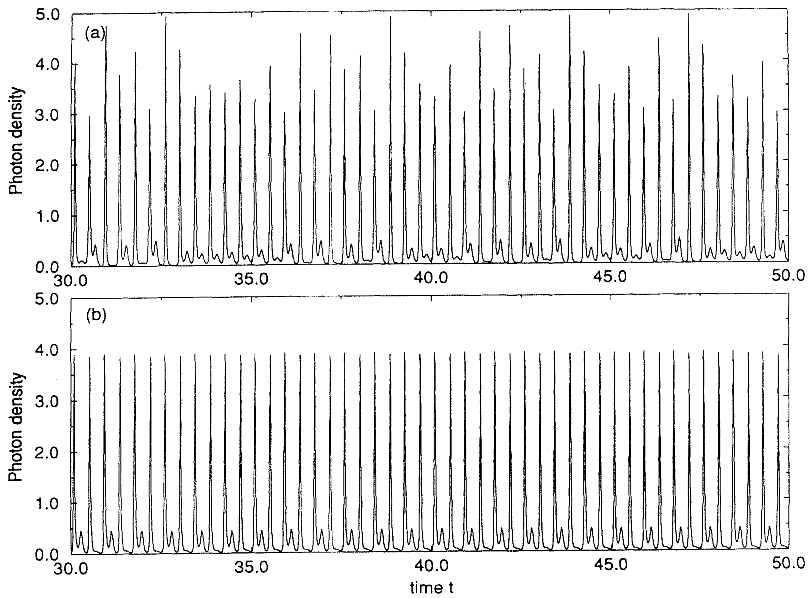

FIG. 8. (a) A typical chaotic time series with parameters $I_m/I_{th}=0.55$, $I_b/I_{th}=1.5$, and $f_m=0.8$ GHz, at $r_1=r_2=0$; (b) tamed periodic orbit at $r_1=0.076$ and $r_2=0.5$. The time t has been rescaled with $\tau_e=3$ ns.

distribution diagram described above. The paper presented here may be recognized as a simple extension of information theory in the characterization of nonlinear dynamic systems and it should provide an appealing approach in controlling chaos.

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